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V-admissibility, Poincaré group and Sturmian-based vector coherent states

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Abstract

We show that there exists a V-admissible-type subspace in the carrier space of the Wigner representation of the Poincaré group in 1+3 dimensions. This subspace is built up from an orthonormal set of Sturmian-type functions which verify naturally the assumption of rotational invariance under which relativistic coherent states frames were previously obtained. Then, we propose an extension of the concept of V-admissibility for the class of semi-direct product groups, which takes into account the relativity groups which were not covered by the existing approaches. In the light of this extension, the square-integrability modulo an appropriate subgroup of the Wigner representation of the group of Poincaré is revisited and some physical implications are discussed.

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1. Introduction

The theory of coherent states (CS) has been one of the most successful theories at the end of the last century. It has been widely used and has proven so useful in so many different scientific applications (see [1] and the references therein) that it is still attracting attention. One of the reasons for the success of CS is, in quantum mechanics for instance, their close relationship to classical states that allows for a classical reading in quantum situations.

Since its discovery, the concept has naturally evolved; it has been studied in a wide range of subject areas. However, the different approaches known in the literature have been unified, at least as far as concerns the continuous theory, through an interpretation in terms of group theory. There are now some attempts for the construction of a sufficiently broad theory based on semi-groups whose main aim is to encompass the continuous case and the known multi-resolution analysis (MRA) as particular cases [2].

Regarding the continuous case, it is well known today that CS, in general, are, in the simplest cases, elements of the orbit of a nonzero element η in a separable Hilbert space $\mathfrak{H}$

which carries a unitary irreducible representation (UIR) U of a symmetry group G. The key property one needs this representation to have is to be square-integrable on the whole group, or modulo an appropriate closed subgroup H of G, when the first requirement fails. In the latter case, instead of being labelled by elements of G, the CS are labelled by elements of the factor space $X = G/H$ via a (global) Borelian section $\sigma : X \to G$. The square-integrability condition reads [1, 3, 4]

$$\int_G |\langle U(g)\eta|\eta\rangle_{\mathfrak{H}}|^2 \, \mathrm{d}\mu(g) < \infty \tag{1}$$

in the first case, and

$$\int_X |\langle U(\sigma(x))\eta|\eta\rangle_{\mathfrak{H}}|^2 \, \mathrm{d}\nu(x) < \infty \tag{2}$$

in the second case, for some η in $\mathfrak{H}$, with μ being a Haar measure of G, and ν a G-invariant measure on X.

In this paper, we will focus again on the square-integrability of the representation U, especially in the second case where the label space is a homogeneous space. Indeed, this case encompasses the first case where the group elements label the CS and which is recovered when H is reduced to the identity element of G and $\sigma = \mathrm{Id}_G$. The pretext for us [5] comes from the search for a class of *suitable* functions that could be used as basis functions in the construction of vector coherent states (VCS) for the Poincaré group $\mathcal{P}_+^\uparrow(1, 3) = \mathbb{R}^4_{1,3} \rtimes SL(2, \mathbb{C})$, the basic invariance group in quantum mechanics briefly described in section 3. Here, *suitable* means for us functions which can allow one to get rid of (or at least to weaken) the strong admissibility condition that the construction of Poincaré CS is constrained to. These functions that we build up in section 4 are 'copies' (in a sense which will become explicit later) of the so-called *atomic CS* [6]. They belong to (closed) subspaces of the carrier space $\mathfrak{H}^s_w$ of the Wigner (massive) representation of the group. By the way, it appears that, using the above (rotationally invariant) subspaces and borrowing ideas from the concept of V-admissibility proposed by Ali [6], it is possible to give an alternative description of the square-integrability modulo $T \times K$ of U^s_w, where $T \simeq \mathbb{R}$ is the subgroup of time translations of $\mathcal{P}_+^\uparrow(1, 3)$, and $K = SU(2)$. This is done in section 6, and it suggests to go beyond Ali's definition of the V-admissibility in order to capture situations such as those of the Poincaré group and other relativity groups which are square-integrable modulo a closed subgroup with a infinite-dimensional UIR, and which do not fall under the scope of the known notions of α- or V-admissibility [1].

Besides the technical gap the following development fills in the literature, and the new alternative it provides in evaluating the square-integrability of the representations of the alluded relativity groups, another physical application is the possibility it offers in defining a geometric quantization schema on the positive mass shell. This is the subject of a paper in preparation.

It can also be pointed out that, when using basis expansions in calculating spectra of atoms and ions, or in measuring information of the quantum mechanical states of a hydrogenic systems, the choice of the basis is of critical importance [7, 8]. To this end, the Sturmian-related VCS we construct in this paper could be of great interest.

In what follows, without further qualification, G will denote a locally compact topological group, and the notations $\mathbf{u} = \underline{u} = (u_1, u_2, u_3)$ and $u = (u_0, \mathbf{u}) = (u_0, \underline{u})$ will represent, respectively, a 3-vector in $\mathbb{R}^3$ and a 4-vector in $\mathbb{R}^4_{1,3}$, the Minkowski space with signature $(+, -, -, -)$.

2. The α- and the V-admissibility

In order to be as complete as possible, let us say a word on the α-admissibility which stands for a first step in the generalization of the classical notion of square-integrability. It has been used to accommodate the Gilmore and Perelomov approaches [9–11].

Definition 2.1 (α-admissibility)**.** *Let $\alpha : H \to \mathbb{C}$ be a unitary character of the closed subgroup H of G, that is, a map such that*

$$\alpha(h_1h_2) = \alpha(h_1)\alpha(h_2) \qquad \text{and} \qquad |\alpha(h)| = 1, \quad \forall h_1, h_2, h \in H. \tag{3}$$

Let U be a UIR of G in $\mathfrak{H}$, and let $\eta \in \mathfrak{H}$ be a nonzero vector satisfying

$$U(h)\eta = \alpha(h)\eta, \qquad \forall h \in H. \tag{4}$$

η is α-admissible for the representation U if, $\sigma : X \to G$ being a Borelian section,

$$I_\alpha(\eta) = \int_X |\langle U(\sigma(x))\eta|\eta\rangle|^2 \,\mathrm{d}\nu(x) < \infty. \tag{5}$$

It is important to point out that this definition does not depend on the choice of the section, and it can be shown easily that, if α_1 and α_2 are different unitary characters, and if two vectors η_1 and η_2 are, respectively, α_1- and α_2-admissible, then, for $h \in H$, η_1 and η_2 are mutually orthogonal as eigenvectors of the operator $U(h)$ associated with different eigenvalues, and so are the corresponding eigenspaces $\mathfrak{H}^{\alpha_1}$ and $\mathfrak{H}^{\alpha_2}$.

Let $\mathcal{A}_\alpha \neq \{0\}$ denote the set of all α-admissible vectors in $\mathfrak{H}$. For any $\eta \in \mathcal{A}_\alpha$, the representation U is square-integrable modulo H, and, on $\mathfrak{H}$, the resolution of the identity

$$[c_\alpha(\eta)]^{-1} \int_X |\eta_{\sigma(x)}\rangle\langle\eta_{\sigma(x)}| \,\mathrm{d}\nu(x) = \mathbb{I}_{\mathfrak{H}} \tag{6}$$

holds, where

$$c_\alpha(\eta) = \frac{I_\alpha(\eta)}{\|\eta\|^2} \qquad \text{and} \qquad \eta_{\sigma(x)} = U(\sigma(x))\eta. \tag{7}$$

The mathematical and physical consequences of equation (6) are extensively discussed in [1, 11]. Replacing the unitary character α by a finite-dimensional representation of H, one obtains a more general framework for the construction of VCS [1, 12, 13].

Definition 2.2 (V-admissibility)**.** *Under the same considerations as above, let V be a finite-dimensional UIR of H. A subspace $\mathfrak{K}$ of $\mathfrak{H}$ is said to be V-admissible if the restriction $U(H) \mid \mathfrak{K}$ of $U(H)$ to $\mathfrak{K}$ is unitarily equivalent to V, and if there exists a nonzero vector $\eta \in \mathfrak{K}$ such that*

$$I(\eta) = \int_X \langle\eta|\mathbb{P}_{\mathfrak{K}}([g])\eta\rangle \,\mathrm{d}\nu([g]) < \infty, \tag{8}$$

where $\mathbb{P}_{\mathfrak{K}}$ is the projection on $\mathfrak{K}$, and $\mathbb{P}_{\mathfrak{K}}([g]) = U(g)\mathbb{P}_{\mathfrak{K}}U(g)^\dagger$. If $\mathfrak{K}$ exists, U is said square-integrable modulo H.

To shed light on the important implications of this new definition, consider the subset $\mathcal{D} \subset \mathfrak{H}$ spanned by the vectors $U(g)\eta$, $\eta \in \mathfrak{K}$, $g \in G$, and define the map

$$W_\eta : \mathcal{D} \to L^2_{\mathfrak{K}}(X, \mathrm{d}\nu) \qquad \phi \mapsto W_\eta\phi \tag{9}$$

by

$$(W_\eta\phi)([g]) = [c_V(\eta)]^{-1/2}\mathbb{P}_{\mathfrak{K}}U(g^{-1})\phi, \tag{10}$$

where $c_V(\eta) = \frac{I(\eta)}{\|\eta\|^2}$. The following theorem is obtained, which generalizes the results obtained in the α-admissibility context.

Theorem 2.3. *Let G be a locally compact group, H a closed subgroup of G, U a UIR of G on the Hilbert space $\mathfrak{H}$, and V a finite-dimensional UIR of H. Let $\mathfrak{K}$ be a V-admissible subspace of $\mathfrak{H}$. Then, the mapping (9) extends to all $\mathfrak{H}$ as a linear isometry, so that its range, $\mathfrak{H}_\eta$, is a closed subspace of $L^2_{\mathfrak{K}}(X, \mathrm{d}\nu)$. On $\mathfrak{H}$, the resolution of the identity*

$$[c_V(\eta)]^{-1} \int_X \mathbb{P}_{\mathfrak{K}}([g])\, \mathrm{d}\nu([g]) = \mathbb{I}_{\mathfrak{H}} \tag{11}$$

holds. The subspace $\mathfrak{H}_\eta = W_\eta \mathfrak{H} \subset L^2_{\mathfrak{K}}(X, \mathrm{d}\nu)$ is a reproducing kernel Hilbert space; the corresponding projection operator

$$\mathbb{P}_\eta = W_\eta W_\eta^* \qquad \left(\mathbb{P}_\eta L^2_{\mathfrak{K}}(X, \mathrm{d}\nu) = \mathfrak{H}_\eta\right) \tag{12}$$

has the reproducing kernel K_η,

$$K_\eta(g, g') = [c_V(\eta)]^{-1} \mathbb{P}_{\mathfrak{K}} U(g^{-1}) U(g') \mathbb{P}_{\mathfrak{K}}$$
$$\Leftrightarrow \quad \int_X K_\eta(g, g'') K_\eta(g'', g')\, \mathrm{d}\nu([g'']) = K_\eta(g, g'), \tag{13}$$

as its integral kernel, that is,

$$(\mathbb{P}_\eta \Phi)(g) = \int_X K_\eta(g, g') \Phi(g')\, \mathrm{d}\nu([g']), \qquad \Phi \in L^2_{\mathfrak{K}}(X, \mathrm{d}\nu). \tag{14}$$

Furthermore, W_η intertwines U and the induced representation ${}^V U : G \to L^2_{\mathfrak{K}}(X, \mathrm{d}\nu)$ defined by

$$({}^V U(g)\mathbf{f})([g']) = \mathbf{f}(g^{-1}[g']), \tag{15}$$

say,

$$W_\eta U(g) = {}^V U(g) W_\eta, \qquad g \in G. \tag{16}$$

If we choose $H = \{e\}$, the classical notion of admissibility (on the whole group), the α-, and the V-admissibility agree. It is also worth mentioning that the conditions stated in definitions 2.1 and 2.2 are naturally fulfilled if H is, for instance, Abelian or compact.

A nice geometrical description of VCS has been carried out by Ali [6], with an application to the isochronous group of Galilei, which integrates perfectly this definition of the V-admissibility. In that description, $\mathfrak{K}$ is considered as the carrier space of V and, as such, is of finite dimension. Therefore, this description does not apply to groups such as the Poincaré group for which VCS have been constructed however. The reason of this failure in encompassing such a group is that the restriction of U to the subgroup H used is of infinite dimension. In fact, the subspace $\mathfrak{K}$ does not need to carry the representation V of H [12, 13]. However, we show, using the example of the Poincaré group we work out in this paper, that the same ideas apply as well, and that there is a way to extend Ali's description.

The interest of extending Ali's result relies also on the fact that, in physical concrete situations, X is given *a priori*, and, when it is realized as a homogeneous space of a symmetry group G by an appropriate subgroup H, the latter is, in many cases, of the form $H = \mathbb{W} \rtimes S_0$, where $\mathbb{W}$ is a vector space, and S_0 is a closed subgroup of G. Therefore, the representation of H is not necessarily of finite dimension.

Subsequently, there is another question. When the square-integrability in the classical sense (1) of the representation of the group fails, which H, V, $\mathfrak{K}$ do we use in order to recover a notion of square-integrability in the sense of the V-admissibility or, at least, in the sense of the formula (2)? Again, for the group of Poincaré (and it seems that the scheme works likewise for numerous groups with no square-integrable representation), the answer for H is known [14]. Intuitively, and like in almost all the known examples, the convenient H is taken to be the subgroup corresponding to the parameters that are responsible for the divergence of the integral (1), or to make the action of G on the particular orbit used not free. Hence, proceeding from the space G/H instead from the entire group amounts to cancelling out those undesirable parameters. For V and $\mathfrak{K}$, we have got a precise response for the Poincaré group while examining the important question of derivation of *good* fiducial functions for VCS construction.

3. Poincaré coherent states

In this paper, the Poincaré group is the semi-direct product group $G = \mathcal{P}_+^\uparrow(1,3) = \mathbb{R}^4_{1,3} \rtimes SL(2,\mathbb{C})$. It is the universal double covering of the inhomogeneous Lorentz group $\mathbb{R}^4_{1,3} \rtimes SO_0(1,3)$, where $SO_0(1,3)$ is the proper orthochronous Lorentz group. This covering group is usually used to take into account the description of half integral spin particles in addition to the integral ones that the inhomogeneous Lorentz group only describes. The group law is given by

$$(a, A)(a', A') = (a + L(A)a', AA'), \qquad (a, A), (a', A') \in \mathcal{P}_+^\uparrow(1,3), \tag{17}$$

where

$$\begin{aligned} &L : SL(2,\mathbb{C}) \to SO_0(1,3) \\ &A \mapsto L(A) = \left(L(A)^\mu_\nu\right)^3_{\mu,\nu=0} = \left(\tfrac{1}{2}\,\mathrm{Tr}(A\sigma_\nu A^\dagger \sigma^\mu)\right)^3_{\mu,\nu=0} \end{aligned} \tag{18}$$

is the fundamental homomorphism, $\sigma_0 = \mathbb{I}_2$ is the identity matrix in two dimensions, and the σ_i, $i = 1, 2, 3$, are the well-known Pauli matrices, that is,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \text{and} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{19}$$

The Wigner representation $U^s_w = \mathrm{Ind}^G_K \mathcal{D}^s$ of the group, induced, for $s \in \mathbb{N}/2$, from the $(2s{+}1)$-dimensional spinorial UIR $\mathcal{D}^s$ of $K = SU(2)$, is given by

$$\left[U^s_W(a, A)\phi\right](k) = \exp\{\mathrm{i}\langle k, a\rangle\}\mathcal{D}^s(w(k, A))\phi(L(A)^{-1}k), \tag{20}$$

with

$$w(k, A) = S(k)^{-1} A S(L(A)^{-1}k) \in SU(2),$$

where

- $\langle k, a\rangle = k \cdot a = k_0 a_0 - \sum_{j=1}^3 k_j a_j$ defines the duality map $\widehat{\mathbb{R}}^4_{1,3} \times \mathbb{R}^4_{1,3} \to \mathbb{R}$, with $\widehat{\mathbb{R}}^4_{1,3} \simeq \mathbb{R}^4_{1,3}$;
- $\phi \in \mathfrak{H}^s_w = \mathbb{C}^{2s+1} \otimes L^2\big(\mathcal{V}^+_m, k_0^{-1}\,\mathrm{d}\mathbf{k}\big)$, the carrier space of the representation.
- $k_0^{-1}\,\mathrm{d}\mathbf{k}$ is the Haar measure on the well-known positive mass shell

  $$\mathcal{V}^+_m = \left\{k = (k_0, \mathbf{k}) \in \mathbb{R}^4 : k^2 = k_0^2 - \|\mathbf{k}\|^2 = m^2\right\} \simeq SL(2,\mathbb{C})/K, \qquad m > 0.$$
- The map

  $$\begin{aligned} &S : \mathcal{V}^+_m \to SL(2,\mathbb{C}) \\ &k \mapsto S(k) = \frac{m\mathbb{I}_2 + \sigma \cdot \overline{k}}{\sqrt{m(k_0 + m)}}, \end{aligned} \tag{21}$$

  with

  $$\overline{k} = (k_0, -\mathbf{k}), \qquad \sigma = (\sigma_\mu)^3_{\mu=0}, \qquad \sigma \cdot \overline{k} = \sum_{\mu=0}^3 k_\mu \sigma_\mu,$$

gives *the* image in $SL(2, \mathbb{C})$ of *the* Lorentz boost $\Lambda_k \in SO_0(1, 3)$, associated with $k \in \mathcal{V}_m^+$, which takes the base point $\widetilde{m} = (m, \mathbf{0}) \in \mathcal{V}_m^+$ to k, say, $\Lambda_k \tilde{m} = k$. The boost is explicitly given by

$$\Lambda_k = \frac{1}{m}\begin{pmatrix} k_0 & \mathbf{k}^\dagger \\ \mathbf{k} & mV_k \end{pmatrix} = \Lambda_k^\dagger, \qquad \text{with} \quad V_k = \mathbb{I}_3 + \frac{\mathbf{k}\otimes\mathbf{k}^\dagger}{m(k_0+m)} = V_k^\dagger = V_{\overline{k}}. \tag{22}$$

The representation U_w^s is not square-integrable modulo $H = T \times K$. It is also important to mention that K is the little group of the point $\widetilde{m}$, and the maximal compact subgroup that appears in the Cartan decomposition, and the Iwasawa NAK-factorization of $SL(2, \mathbb{C})(N \simeq \mathbb{C}$, and $A \simeq SO_0(1, 1) \simeq \mathbb{R})$ [17]. These structures play an important role in the construction of CS associated with the group $\mathcal{P}_+^\uparrow(1, 3)$:

- They induce an interesting property by which any Lorentz transformation $\Lambda(p)$, parametrized by $p \in \mathcal{V}_m^+$, can be decomposed into a rotation $\rho(p) = \begin{pmatrix} 1 & \mathbf{0}^\dagger \\ \mathbf{0} & \rho(\mathbf{p}) \end{pmatrix} \in SO_0(1, 3)$, where $\rho(\mathbf{p}) \in SO(3)$ is an axial rotation that maps the z-axis on the direction of the vector $\mathbf{p}$, followed by a pure boost in the direction of p, that is, $\Lambda(p) = \Lambda_p \rho(p)$. Moreover, due to the homeomorphism $\mathcal{V}_m^+ \simeq NA$, we have that any element $B \in SL(2, \mathbb{C})\backslash K \simeq NA$ can be uniquely associated with a pure boost Λ_p, with $p \in \mathcal{V}_m^+$.
- They give rise to very useful sections $\sigma : \Gamma = G/(T \times K) \to G$. The Poincaré CS are labelled by pairs $(\mathbf{q}, \mathbf{p}) \in \Gamma = G/(T\times K) \simeq \mathbb{R}^3 \times \mathcal{V}_m^+ \simeq \mathbb{R}^3 \times \mathbb{R}^3$ of position–momentum coordinates via *affine* sections $\sigma = \sigma_0 \cdot f$ defined by

$$\sigma(\mathbf{q}, \mathbf{p}) = \sigma_0(\mathbf{q}, \mathbf{p})((f(\mathbf{q}, \mathbf{p}), \mathbf{0}), R(\mathbf{p})) = (\widehat{q}, S(p)R(\mathbf{p})), \qquad \widehat{q} = (\widehat{q}_0, \widehat{\mathbf{q}}), \tag{23}$$

where

$$\sigma_0(\mathbf{q}, \mathbf{p}) = ((0, \mathbf{q}), S(p)) \tag{24}$$

determines the basic section used, and $f : \Gamma \to \mathbb{R}$ is a measurable function given by

$$f(\mathbf{q}, \mathbf{p}) = \varphi(\mathbf{p}) + \mathbf{q}\cdot\boldsymbol{\vartheta}(\mathbf{p}) = \varphi(\mathbf{p}) + \mathbf{q}\cdot\left(\frac{\mathbf{p}}{m} - V_p\boldsymbol{\beta}^*(\mathbf{p})\right). \tag{25}$$

$\boldsymbol{\vartheta}$ (respectively R) : $\mathbb{R}^3 \to \mathbb{R}^3$(respectively$SU(2)$) are measurable functions, $\varphi : \mathbb{R}^3 \to \mathbb{R}$ is a kind of gauge-fixing (measurable) function which is set to zero because of its irrelevance in the construction. The vector-valued function $\boldsymbol{\beta}^*(\mathbf{p})$ has the dimension of a velocity, and characterizes the section σ. For technical reasons, it is assumed that

$$\|\boldsymbol{\beta}^*(\mathbf{p})\| < 1 \quad \Leftrightarrow \quad \widehat{q}^2 < 0. \tag{26}$$

Such a section σ is then called a *space-like affine admissible section*.

Therefore, the Poincaré coherent states are the functions

$$\boldsymbol{\eta}^l_{\sigma(\mathbf{q},\mathbf{p})} = U_w^s(\sigma(\mathbf{q}, \mathbf{p}))\boldsymbol{\eta}^l, \qquad (\mathbf{q}, \mathbf{p}) \in \Gamma, \quad l = 1, 2, \ldots, 2s+1, \tag{27}$$

for a given set of elements $\boldsymbol{\eta}^l$ of $\mathfrak{H}_w^s$ that satisfy the condition of (global) invariance under rotation

$$\mathcal{D}^s(R)\left(\sum_{l=1}^{2s+1}|\boldsymbol{\eta}^l\rangle\langle\boldsymbol{\eta}^l|\right)\mathcal{D}^s(R)^\dagger = \sum_{l=1}^{2s+1}|\boldsymbol{\eta}^l\rangle\langle\boldsymbol{\eta}^l|, \qquad \forall R \in SU(2). \tag{28}$$

The energy operator P_0 is defined on $\mathfrak{H}_w^s$ by

$$(P_0\boldsymbol{\eta})(k) = k_0\boldsymbol{\eta}(k), \qquad \forall k \in \mathcal{V}_m^+, \tag{29}$$

with $(\widehat{\mathbf{e}}^l)_{l=1}^{2s+1}$ being the canonical basis of $\mathbb{C}^{2s+1}$. The condition (28) is fulfilled when the $\boldsymbol{\eta}^l$ are taken to be of the form $\boldsymbol{\eta}^l = \widehat{\mathbf{e}}^l \otimes \eta$, and $\eta \in \mathcal{D}(P_0) \cap \mathfrak{H}_w^0$ is assumed, for explicit calculation purposes, to have a rotation-invariant square modulus, say,

$$|\eta(\rho k)|^2 = |\eta(k)|^2, \qquad \forall \rho \in SO(3). \tag{30}$$

The two conditions (28) and (30) are referred to as the *assumption of rotational invariance* [14], and we will denote them by ARI.

In what follows, we will construct a concrete example of basis functions which satisfy naturally the ARI.

4. A class of relativistic Sturmians

We use here, in the context of the Poincaré group, a model developed for a non-relativistic system of free spinning particles whose symmetry is described by the isochronous group of Galilei. The Hilbert space $\mathfrak{H}_w^s$ involves globally K-invariant finite-dimensional subspaces, pairwise orthogonal. Picking up a basis of one properly chosen of these spaces, one obtains a system of relativistic Sturmian functions which generates a finite rank continuous frame of Poincaré CS [4].

Let us describe the construction.

Consider the operator

$$\mathcal{F} : \mathfrak{H}_w^s = \mathbb{C}^{2s+1} \otimes L^2\left(\mathcal{V}_m^+, k_0^{-1}\,\mathrm{d}\mathbf{k}\right) \to \mathcal{H}^s = \mathbb{C}^{2s+1} \otimes L^2(\mathbb{R}^3, \mathrm{d}\mathbf{k}) \tag{31}$$

defined by

$$(\mathcal{F}\boldsymbol{\eta})(\mathbf{k}) = k_0^{-1/2}\boldsymbol{\eta}(k), \qquad \forall \mathbf{k} \in \mathbb{R}^3, \tag{32}$$

where $k_0 = \|\mathbf{k}\|^2 + m^2$ and $k = (k_0, \mathbf{k}) \in \mathcal{V}_m^+$. $\mathcal{F}$ is obviously a unitary operator. Furthermore, note that, S^2 denoting the unit sphere of $\mathbb{R}^3$, any vector $\mathbf{k}$ of $\mathbb{R}^3$ can be written as

$$\mathbf{k} = \|\mathbf{k}\|\widehat{\mathbf{k}}, \qquad \text{with} \qquad \widehat{\mathbf{k}} = \mathbf{k}(\theta, \varphi), \quad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi). \tag{33}$$

Setting $r = \|\mathbf{k}\|$ we can write that

$$L^2(\mathbb{R}^3, \mathrm{d}\mathbf{k}) \simeq L^2(\mathbb{R}_+, r^2\,\mathrm{d}r) \otimes L^2(S^2, \mathrm{d}\Omega(\widehat{\mathbf{k}})). \tag{34}$$

Using an orthonormal basis $\{e_\ell\}_{\ell=1}^\infty$ of $L^2(\mathbb{R}_+, r^2\,\mathrm{d}r)$, the spherical harmonics $\{Y^{\mu\nu} : \mu \in \mathbb{N}, \nu = -\mu, -\mu + 1, \ldots, \mu - 1, \mu\}$, and the Wigner decomposition of rotations, we can decompose $\mathcal{H}^s$ as

$$\mathcal{H}^s \simeq \bigoplus_{\ell=1}^{\infty} \bigoplus_{\mu=0}^{\infty} \bigoplus_{J=|s-\mu|}^{s+\mu} \mathcal{H}_\ell^{\mu J}, \tag{35}$$

where $\mathcal{H}_\ell^{\mu J}$ is a (2J+1)-dimensional space which carries a representation unitarily equivalent to the representation $\mathcal{D}^J$ of K. Let $(\widehat{\mathbf{e}}_l)_{l=1}^{2s+1}$ be the canonical basis of $\mathbb{C}^{2s+1}$, and define $(\widehat{\mathbf{e}}^{sj})_{j=1}^{2s+1} = (\delta_{s+j+1,l}\widehat{\mathbf{e}}_l)_{j=1}^{2s+1}$. The functions $\mathbf{f}_\ell^{\mu JM}$ defined by

$$\mathbf{f}_\ell^{\mu JM}(\mathbf{k}) = \sum_j C(s, j; \mu, M - j|JM)\widehat{\mathbf{e}}^{sj} \otimes Y^{\mu M-j}(\widehat{\mathbf{k}})e_\ell(\|\mathbf{k}\|) \tag{36}$$

form an orthonormal basis of $\mathcal{H}_\ell^{\mu J}$. The symbols $C(s, j; \mu, M - j|JM)$ denote the Clebsch–Gordan coefficient of $SU(2)(j \in \{-|s - \mu|, -|s - \mu| + 1, \ldots, s + \mu - 1, s + \mu\}$ and $-s \leqslant j \leqslant s, -\mu \leqslant M - j \leqslant \mu$, and $-J \leqslant M \leqslant J)$ [15, 16].

In the context of the (extended) Galilei group, these vectors can be considered as quantum mechanical wavefunctions for a system with spin s, orbital angular momentum μ, and total angular momentum J [6].

We use this decomposition to define in $\mathfrak{H}_w^s$ the subspaces

$$\mathfrak{K}_\ell^{\mu J} = \mathcal{F}^{-1}\mathcal{H}_\ell^{\mu J}, \tag{37}$$

for ℓ, μ, J kept fixed, and it follows from the unitarity of $\mathcal{F}$ that

$$\mathfrak{H}_w^s \simeq \bigoplus_{\ell=1}^{\infty}\bigoplus_{\mu=0}^{\infty}\bigoplus_{J=|s-\mu|}^{s+\mu} \mathfrak{K}_\ell^{\mu J}. \tag{38}$$

The space $\mathfrak{K}_\ell^{\mu J}$ is also of finite dimension. It carries a representation of K which is unitarily equivalent to $\mathcal{D}^J$. This will be made precise later on. An orthonormal basis of $\mathfrak{K}_\ell^{\mu J}$ is given by the set of the functions defined by

$$\eta_\ell^{\mu JM} = \mathcal{F}^{-1}\mathbf{f}_\ell^{\mu JM}, \qquad M = -J, -J+1, \ldots, J-1, J. \tag{39}$$

Choosing then the e_ℓ to be, for instance, the radial Sturmian functions (or generalized Laguerre) defined by

$$e_\ell(r) = e_\ell^\mu(r) = R_{\ell,\mu}(r) = C_{\ell,\mu}\left(\frac{2r}{\ell}\right)^\mu \exp\left(-\frac{r}{\ell}\right) L_{\ell+\mu}^{2\mu+1}\left(\frac{2r}{\ell}\right), \tag{40}$$

we get, with the functions in (39), an orthonormal set of functions eligible (since they naturally verify the ARI) as fiducial functions in the construction of Poincaré VCS. $C_{\ell,\mu}$ denotes a constant of normalization.

We shall show that, for a properly chosen representation V^J of $T \times K$, the subspace $\mathfrak{H}_\ell^{\mu J} = \text{span}\{V^J(T\times K)\mathfrak{K}_\ell^{\mu J}\}$ is V^J-admissible in the carrier space $\mathfrak{H}_w^s$ of the Wigner representation U_w^s of the group. This is surprising, because, due to the effect of time translations, the representation V^J is infinite dimensional, and one could think that $\mathfrak{H}_\ell^{\mu J}$ will fill up $\mathfrak{H}_w^s$. In fact, this is not the case. It then becomes obvious that the framework of the definition 2.2 of the V-admissibility has to be enlarged in order to take into account this new situation. This is done in the next section for the class of semi-direct product groups.

5. The extended V-admissibility

Let $G = \mathbb{V} \rtimes S$ be a semi-direct product group, where $\mathbb{V}$ is an n-dimensional vector space, and $S \subset GL(\mathbb{V}, n)$. Let $\mathbb{W}$ be an m-dimensional vector subspace of $\mathbb{V}$ and S_0 be a closed subspace of S such that $H = \mathbb{W} \rtimes S_0$ is a subgroup of G. Form the quotient space $X = G/H$, and assume that it is endowed with an invariant measure μ under the action of the group G. Consider a Borelian section $\sigma : X \to G$, and let U be a UIR of G on a separable Hilbert space $\mathfrak{H}$. Let $\mathcal{K}$ be a finite-dimensional subspace of $\mathfrak{H}$ (dim $\mathcal{K} = d$, $d \in \mathbb{N}^*$, and $d < \infty$), which is stable under U when the latter is restricted to S_0. Consider a unitary representation V of H such that both restrictions of U and V to S_0 are unitarily equivalent. Take an orthogonal projection $\mathbb{P}$ of $\mathfrak{H}$ onto the subspace $\mathcal{K}$, and define $\mathcal{H} = \text{span}\{V(H)\mathcal{K}\}$. Assume that the representation $h \mapsto V(h)$ of the subgroup H acts irreducibly on $\mathcal{H}$. For each $x \in X$, the sets $\{\eta_{\sigma(x)}^j = U(\sigma(x))\eta^j : j = 1, 2, \ldots, d\}$ are vector coherent states, and the set of vectors

$$\mathfrak{S}_\sigma = \left\{\eta_{\sigma(x)}^j : j = 1, 2, \ldots, d; x \in X\right\} \tag{41}$$

is total in $\mathfrak{H}$.

Using the principal fibre bundle $\pi : G \to X$, we consider the G-homogeneous associated bundle $\pi' : B = G \times_{V(H)} \mathcal{H} \to X$ also fibred over X with structure group H [18]. It is obtained by identifying elements $(g, \eta), (g', \eta') \in G \times \mathcal{H}$, whenever $g' = gh$ and $\eta' = V(h)^{-1}\eta$, for some $h \in H$. Elements in B can therefore be put into one-to-one correspondence with *coherent vectors* $U(g)\eta \in \mathfrak{H}$, for all $g \in G$ and $\eta \in \mathcal{H}$. The canonical projection π' is then given by

$$\pi'(g, \eta) = \pi'(U(g)\eta) = [g], \tag{42}$$

while sections in B have the form

$$\sigma' : x \in X \mapsto \sigma'(x) = (\sigma(x), f(g)) \in B, \qquad (x = [g]), \tag{43}$$

where $f : G \to \mathcal{H}$ is a function satisfying

$$f(gh) = V(h)^{-1} f(g), \qquad h \in H. \tag{44}$$

Sections in the bundle B can then be put into one-to-one correspondence with such functions.

Consider next the functions $\widetilde{\Phi} : G \to \mathcal{H}$, defined by

$$\widetilde{\Phi}(g) = V(h)^{-1}\mathbb{P}U(\sigma([g])^{-1})\phi, \qquad \phi \in \mathfrak{H}, \quad g = \sigma([g])h. \tag{45}$$

It can be easily seen that these functions satisfy (44), and also describe sections of B. For each $g \in G$ we have $\widetilde{\Phi}(g) \sim (h, \mathbb{P}U(\sigma([g])^{-1})\phi)$. On the other hand, each $\widetilde{\Phi}$ induces an (algebraic) isomorphism $\widetilde{\widetilde{\Phi}} : X \to \mathrm{Ran}(\widetilde{\Phi})$. As a set, $\mathrm{Ran}(\widetilde{\Phi}) \simeq H \times \mathcal{K}$. Considering the projection $\mathrm{pr}_{\mathcal{K}} : H \times \mathcal{K} \to \mathcal{K}$, $(h, \eta) \mapsto \eta$, define the functions $\Phi : X \to \mathcal{K}$ by

$$\Phi(x) = \mathrm{pr}_{\mathcal{K}} \circ \widetilde{\widetilde{\Phi}}(x) = \mathbb{P}U(\sigma(x)^{-1})\phi, \qquad x \in X, \tag{46}$$

and form the set $\mathfrak{H}_K$ of such functions. The following diagram gives a global view of the maps considered in the description:

$$\begin{tikzcd}
G \arrow[r, "\widetilde{\Phi}"] \arrow[d, "\pi"'] & B = G \times_{V(H)} \mathcal{H} \arrow[r, "\mathrm{pr}_{\mathcal{H}}"] & \mathcal{H} \\
X = G/H \arrow[r, "\widetilde{\widetilde{\Phi}}"] \arrow[rr, bend right, "\Phi"'] & \mathrm{Ran}(\widetilde{\Phi}) \simeq H \times \mathcal{K} \arrow[u, "\mathrm{id}_G \otimes V(H)"] \arrow[r, "\mathrm{pr}_{\mathcal{K}}"] & \mathcal{K} \arrow[u, "V(H)"']
\end{tikzcd}$$

For practical reasons (in fact, in order to realize $\mathfrak{H}_K$ as a space of L^2 functions), it is important to look at under what conditions $\mathfrak{H}_K$ is a subspace of $L^2_{\mathcal{K}}(X, \mathrm{d}\mu)$. The square-integrability of the sections Φ would imply that

$$\begin{aligned} &\int_X \langle\phi|\mathbb{P}_\sigma(x)\phi\rangle \,\mathrm{d}\mu(x) = \langle\phi|\mathfrak{A}_\sigma\phi\rangle < \infty, \qquad \forall \phi \in \mathfrak{H}, \\ &\mathbb{P}_\sigma(x) = U(\sigma(x))\mathbb{P}U(\sigma(x))^\dagger. \end{aligned} \tag{47}$$

Therefore, we would have that the self-adjoint and positive operator

$$\int_X \mathbb{P}_\sigma(x)\,\mathrm{d}\mu(x) = \mathfrak{A}_\sigma. \tag{48}$$

is also bounded. Using the spectral theorem for self-adjoint operators and related results in C^*-algebras (see [19] for further details), and denoting the spectrum of the operator $\mathfrak{A}_\sigma$ by $\mathrm{spec}(\mathfrak{A}_\sigma)$, it can be shown that $\mathfrak{A}_\sigma$ is unitarily equivalent to a multiplication operator M_φ on $L^2(\mathrm{spec}(\mathfrak{A}_\sigma))$ endowed with an appropriate measure. From this unitary equivalence, it can be shown that M_φ is strictly positive, thus invertible. It follows that $\mathfrak{A}_\sigma$ is also invertible.

Conversely, it can be seen easily, still from (47), that the boundedness of $\mathfrak{A}_\sigma$ implies that $\mathfrak{H}_K$ is a subspace of $L^2_{\mathcal{K}}(X, \mathrm{d}\mu)$. We can then state the following important result:

Theorem 5.1. *The operator $\mathfrak{A}_\sigma$ is bounded and invertible if and only if $\mathfrak{H}_K \subset L^2_{\mathcal{K}}(X, \mathrm{d}\mu)$.*

Let us mention that in many situations of interest, as in the cases of the Poincaré and Galilei groups, $\mathfrak{A}_\sigma$ is itself (at least under additional particular conditions) a strictly positive multiplication operator, hence invertible. On the other hand, in the situation depicted in [6] for instance, where H is reduced to S_0, $U(g)$ commutes with $\mathfrak{A}_\sigma$ for all g, and, with the invariance of the measure μ and the irreducibility of U, $\mathfrak{A}_\sigma = \lambda\mathbb{I}_{\mathfrak{H}}$, for some $\lambda > 0$.

Assuming that $\mathfrak{A}_\sigma$ is invertible, define the linear map $W_\sigma : \mathfrak{H} \to \mathfrak{H}_K$ by

$$(W_\sigma\phi)(x) = \Phi(x) = \mathbb{P}U(\sigma(x)^{-1})\phi, \qquad \phi \in \mathfrak{H}, \quad x \in X. \tag{49}$$

The set $\mathfrak{H}_K$ has a natural structure of a reproducing kernel Hilbert space with the matrix-valued reproducing kernel $K_\sigma : X \times X \to \mathcal{L}(\mathcal{K})$ defined by

$$K_\sigma(x, x') = \mathbb{P}U(\sigma(x)^{-1})\mathfrak{A}_\sigma^{-1}U(\sigma(x'))\mathbb{P}, \qquad x, x' \in X. \tag{50}$$

We have that, for all $x, x' \in X$, $K_\sigma(x, x')^* = K_\sigma(x', x)$, and, as an operator on $\mathcal{K}$, $K_\sigma(x, x)$ is strictly positive. K_σ is then a strictly positive definite kernel, so it can be used to define a scalar product $\langle\cdot|\cdot\rangle_K$ on the linear span of the functions denoted by $\Phi^\eta_x = K_\sigma(\cdot, x)\eta$ and given by

$$\Phi^\eta_x(x') = K_\sigma(x', x)\eta = \left(W_\sigma\mathfrak{A}_\sigma^{-1}U(\sigma(x))\eta\right)(x'), \tag{51}$$

for all $x, x' \in X$ and $\eta \in \mathcal{K}$. The scalar product reads

$$\left\langle\Phi^\eta_x\middle|\Phi'^{\eta'}_{x'}\right\rangle_K = \langle\eta|K_\sigma(x, x')\eta'\rangle_{\mathfrak{H}}, \tag{52}$$

and extends to $\mathfrak{H}_K$. Endowed with this scalar product, $\mathfrak{H}_K$ is complete, hence a Hilbert space, and the operator W_σ is unitary. An evaluation condition for the functions $\Phi \in \mathfrak{H}_K$ at points $x \in X$ is given by

$$\left\langle\Phi^\eta_x\middle|\Phi\right\rangle_K = \langle K_\sigma(\cdot, x)\eta|\Phi\rangle_K = \left\langle\eta\middle|\left(W_\sigma\mathfrak{A}_\sigma^{-1}W_\sigma^{-1}\Phi\right)(x)\right\rangle_{\mathfrak{H}}, \quad \eta \in \mathfrak{H}. \tag{53}$$

This reflects the fact that $\mathfrak{H}_K$ is a reproducing kernel Hilbert space.

Since now $\mathfrak{H}_K \subset L^2_{\mathcal{K}}(X, \mathrm{d}\mu)$, then, for any Φ, Φ' in $\mathfrak{H}_K$,

$$\int_X \langle\Phi(x)|\Phi'(x)\rangle_{\mathfrak{H}}\,\mathrm{d}\mu(x) < \infty. \tag{54}$$

Moreover, for all $x, x', x'' \in X$, we have that

$$\int_X K_\sigma(x, x'')K_\sigma(x'', x')\,\mathrm{d}\mu(x'') = K_\sigma(x, x''), \tag{55}$$

and this is nothing but a restatement of the square-integrability of U modulo (H, σ). Therefore, we propose the following:

Definition 5.2 (Extended **V**-admissibility)**.** *Let G, $\mathbb{V}$, S, H, S_0, $\mathbb{W}$, X, σ and μ be defined as previously. Let U be a UIR of G on a separable Hilbert space $\mathfrak{H}$, and let V be a UIR of H. A subspace $\mathcal{H}$ of $\mathfrak{H}$ is said to be V-admissible if*

(i) there exists a finite-dimensional subspace $\mathcal{K}$ of $\mathcal{H}$ such that $\mathrm{span}\{V(H)\mathcal{K}\} \subseteq \mathcal{H}$*,*

(ii) the restriction $U(S_0)$ of U to S_0 is unitarily equivalent on $\mathcal{K}$ to the restriction $V(S_0)$ of V to S_0, and

(iii) there exists a nonzero vector $\eta \in$ $\mathrm{span}\{V(H)\mathcal{K}\}$ *such that*

$$I(\eta) = \int_X \langle\eta|\mathbb{P}_\sigma(x)\eta\rangle\,\mathrm{d}\mu(x) < \infty. \tag{56}$$

If such a subspace $\mathcal{H}$ exists, U is said to be square-integrable modulo H.

Let us point out that in the case of the Poincaré group developed in this paper, $\mathbb{W} = \mathbb{R}$ is the time translations subgroup, $S_0 = SU(2)$, and $\mathcal{H} \simeq \mathfrak{H}_\ell^{\mu J}$, for some ℓ, μ, and J. In the case of the isochronous Galilei group worked out in Ali's paper ([6]), $\mathbb{W} = \mathbb{R}$ corresponds to the phase subgroup, and $S_0 = SU(2)$ again. In both cases V is of the form $V = \chi \otimes L$, where χ is a unitary representation of $\mathbb{W}$ and L is a UIR of $SU(2)$, namely, the (2s+1)-dimensional spinorial representation $\mathcal{D}^s$, $s \in \mathbb{N}/2$. Nevertheless, there is a fundamental difference between these two situations: while the phase factor in the Galilean case is one dimensional (that is, a character) and is defined by

$$(\chi(\theta)\phi)(\mathbf{k}) = \mathrm{e}^{\mathrm{i}\theta}\phi(\mathbf{k}), \qquad \mathbf{k} \in \mathcal{O}^* \simeq \widehat{\mathbb{R}}^3, \quad \phi \in \mathcal{H}^s, \tag{57}$$

with $\mathcal{H} \simeq \mathcal{H}_\ell^{\mu J}$, for ℓ, μ and J kept fixed, in the Poincaré case, the unitary representation χ is of infinite dimension and is given by

$$(\chi(t)\phi)(k) = \mathrm{e}^{\mathrm{i}\langle t,k\rangle}\phi(k), \qquad k \in \mathcal{O}^* \simeq \mathcal{V}_m^+, \quad \phi \in \mathfrak{H}_W^s. \tag{58}$$

In the setting of definition 5.2, denote by $\mathcal{D}$ the dense set in $\mathfrak{H}$ spanned by $U(G)\eta$. $\mathcal{D}$ is stable under U, and the map $W_\sigma : \mathcal{D} \to L^2_{\mathcal{K}}(X, \mathrm{d}\mu)$

$$(W_\sigma\phi)(x) = \mathbb{P}U(\sigma(x)^{-1})\phi \tag{59}$$

is well defined. It can be easily shown that W_σ is an isometry. The following theorem then follows directly:

Theorem 5.3. *Let $\mathbb{V}$ be a vector space of dimension n, $S \subset GL(\mathbb{V}, n)$, and $G = \mathbb{V} \rtimes S$. Let $H = \mathbb{W} \rtimes S_0$ be a subgroup of G, $X = G/H$, and μ be an invariant measure on X. Let U be a UIR of G on a separable Hilbert space $\mathfrak{H}$, and let V be a UIR of H. Let $\mathcal{H}$ be a V-admissible subspace of $\mathfrak{H}$. Then,*

(i) The mapping (59) extends to all $\mathfrak{H}$ as a linear isometry. Its range coincides with the set $\mathfrak{H}_K$, which is also closed in the L^2-norm as a subspace of $L^2_{\mathcal{K}}(X, \mathrm{d}\mu)$.

(ii) On $\mathfrak{H}$, the resolution of the operator

$$\int_X \mathbb{P}_\sigma(x)\,\mathrm{d}\mu(x) = \mathfrak{A}_\sigma \tag{60}$$

holds, where $\mathfrak{A}_\sigma$ is a self-adjoint, positive, bounded, invertible operator.

(iii) The subspace $\mathfrak{H}_K = W_\sigma\mathfrak{H} \subset L^2_{\mathcal{K}}(X, \mathrm{d}\mu)$ is a reproducing kernel Hilbert space, where the reproducing kernel K_σ, defined by

$$K_\sigma(x, x') = \mathbb{P}U(\sigma(x)^{-1})\mathfrak{A}_\sigma^{-1}U(\sigma(x'))\mathbb{P}$$
$$\Leftrightarrow \quad \int_X K_\sigma(x, x'')K_\sigma(x'', x')\,\mathrm{d}\mu(x'') = K_\sigma(x, x'), \tag{61}$$

is the integral kernel of the projection operator

$$\mathbb{P}_\sigma = W_\sigma W_\sigma^*, \qquad \mathbb{P}_\sigma L^2_{\mathcal{K}}(X, \mathrm{d}\mu) = \mathfrak{H}_K, \tag{62}$$

that is,

$$(\mathbb{P}_\sigma\Phi)(x) = \int_X K_\sigma(x, x')\Phi(x')\,\mathrm{d}\mu(x'), \qquad \Phi \in L^2_{\mathcal{K}}(X, \mathrm{d}\mu). \tag{63}$$

It is straightforward that the case of V-admissibility discussed here is a generalization of Ali's description where H is reduced to S_0. But it is important to point out that, with this description, we cannot unfortunately compute in general covariance relations of the type (16) for the corresponding coherent states. The main reason is that, in general, $\mathcal{K}$ is not stable under $V(H)$, and the resolution operator $\mathfrak{A}_\sigma$ is not the identity in $\mathfrak{H}$.

6. Application to the group of Poincaré $\mathcal{P}_+^{\uparrow}(1,3)$: the square-integrability modulo $T \times K$ of U_w^s revisited

Let us start by clarifying, for later needs, the action of pure rotations and boosts on the $\boldsymbol{\eta}_\ell^{\mu JM}$ via the representation U_w^s. The proofs of the following two lemmas are in the appendix.

Lemma 6.1. *For all R in K, we have that*

$$U_w^s(0, R)\boldsymbol{\eta}_\ell^{\mu JM} = \mathcal{F}^{-1}\mathcal{D}^J(R)\mathcal{F}\boldsymbol{\eta}_\ell^{\mu JM}, \qquad \text{that is,} \quad \big(U_w^s \,\big|\, K\big) \,\big|\, \mathfrak{K}_\ell^{\mu J} = \mathcal{F}^{-1}\mathcal{D}^J\mathcal{F}. \tag{64}$$

For boosts, we do not get such a symmetric relation as in the case of rotations, because of the different nature of pure rotations and boosts. It is well known that the product of two Lorentz boosts is not in general a boost.

Lemma 6.2. *For all B in $SL(2,\mathbb{C})\backslash K$,*

$$\begin{aligned}\big[U_w^s(0, B)\boldsymbol{\eta}_\ell^{\mu JM}\big](k) &= \big[\mathcal{F}^{-1}\mathcal{D}^J(w(k, B))\mathcal{F}\boldsymbol{\eta}_\ell^{\mu JM}\big](L(w(k, B))L(B)^{-1}k)\\ &= \boldsymbol{\eta}_\ell^{\mu JM}(L(B)^{-1}k), \qquad \forall k \in \mathcal{V}_m^+.\end{aligned} \tag{65}$$

Consider the energy operators P_0 in (29), and the operator $\widetilde{P}_0 : \mathcal{H}^s \to \mathcal{H}^s$ defined by

$$(\widetilde{P}_0\mathbf{f})(\mathbf{k}) = (m^2 + \|\mathbf{k}\|^2)^{1/2}\mathbf{f}(\mathbf{k}), \qquad \forall\, \mathbf{k} \in \mathbb{R}^3. \tag{66}$$

Define the unitary operators

$$\exp(\mathrm{i}P_0) : \mathfrak{H}_w^s \to \mathfrak{H}_w^s \qquad \text{and} \qquad \exp(\mathrm{i}\widetilde{P}_0) : \mathcal{H}^s \to \mathcal{H}^s \tag{67}$$

respectively by

$$[\exp(\mathrm{i}P_0)\boldsymbol{\eta}](k) = \exp(\mathrm{i}k_0)\boldsymbol{\eta}(k), \qquad \boldsymbol{\eta} \in \mathfrak{H}_w^s, \quad k \in \mathcal{V}_m^+, \tag{68}$$

and

$$[\exp(\mathrm{i}\widetilde{P}_0)\mathbf{f}](\mathbf{k}) = \exp(\mathrm{i}\sqrt{m^2 + \|\mathbf{k}\|^2})\mathbf{f}(\mathbf{k}), \qquad \mathbf{f} \in \mathcal{H}^s, \quad \mathbf{k} \in \mathbb{R}^3. \tag{69}$$

$\mathcal{F}$ and $\mathcal{F}^{-1}$ intertwine these operators, that is,

$$\mathcal{F}\exp(\mathrm{i}P_0) = \exp(\mathrm{i}\widetilde{P}_0)\mathcal{F} \quad \Leftrightarrow \quad \mathcal{F}^{-1}\exp(\mathrm{i}\widetilde{P}_0) = \exp(\mathrm{i}P_0)\mathcal{F}^{-1}. \tag{70}$$

Define a representation V^J of $T \times K$ on $\mathfrak{H}_\ell^{\mu J}(J \in \{|s-\mu|, \ldots, s+\mu\} \cap \{0, \ldots, s\})$ by

$$V^J(t, R) = \exp(\mathrm{i}tP_0)\mathcal{F}^{-1}\mathcal{D}^J(R)\mathcal{F}, \qquad t = (t, \mathbb{I}_2) \in T, \quad R \in K. \tag{71}$$

Setting

$$\boldsymbol{\eta}_{\ell,\sigma(\mathbf{q},\mathbf{p})}^{\mu JM} = U_w^s(\sigma(\mathbf{q}, \mathbf{p}))\boldsymbol{\eta}_\ell^{\mu JM}, \qquad (\mathbf{q}, \mathbf{p}) \in \Gamma, \tag{72}$$

we have the following result (proof in the appendix):

Lemma 6.3. *For all $-J \leqslant M', M'' \leqslant J$, we have that*

$$\begin{aligned}c_{V^J}\big(\boldsymbol{\eta}_\ell^{\mu JM'}, \boldsymbol{\eta}_\ell^{\mu JM''}\big) &= \int_\Gamma \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p}\big\langle\boldsymbol{\eta}_\ell^{\mu JM'}\big| \left(\sum_{M=-J}^{J} \big|\boldsymbol{\eta}_{\ell,\sigma(\mathbf{q},\mathbf{p})}^{\mu JM}\big\rangle\big\langle\boldsymbol{\eta}_{\ell,\sigma(\mathbf{q},\mathbf{p})}^{\mu JM}\big|\right) \big|\boldsymbol{\eta}_\ell^{\mu JM''}\big\rangle_{\mathfrak{H}_W^s}\\ &= \big\langle\boldsymbol{\eta}_\ell^{\mu JM'}\big|\mathfrak{A}_{\ell,\sigma}^{\mu J}\boldsymbol{\eta}_\ell^{\mu JM'}\big\rangle_{\mathfrak{H}_W^s},\end{aligned}$$

where $\mathfrak{A}_{\ell,\sigma}^{\mu J}$ is a multiplication operator defined, for $\phi \in \mathfrak{H}_\ell^{\mu J}$, and $k \in \mathcal{V}_m^+$, by

$$\big(\mathfrak{A}_{\ell,\sigma}^{\mu J}\phi\big)(k) = \mathfrak{A}_{\ell,\sigma}^{\mu J}(k)\phi(k), \tag{73}$$

with

$$\mathfrak{A}^{\mu J}_{\ell,\sigma}(k) = (2\pi)^3 m \int_{\mathcal{V}_m^+} \frac{\mathrm{d}\mathbf{p}}{p_0} \big[p_0 + \mathbf{p}^\dagger \rho\big(k \to \Lambda_k^{-1} p\big)^\dagger \beta^*\big(-\underline{\Lambda_k^{-1} p}\big)\big]^{-1}$$
$$\times \sum_{M=-J}^{J} \big|\boldsymbol{\eta}_\ell^{\mu J M}(p)\big\rangle\big\langle \boldsymbol{\eta}_\ell^{\mu J M}(p)\big|$$
$$= (2\pi)^3 m \sum_{M=-J}^{J} \sum_j |C(s, j; \mu, M - j|JM)|^2 \int_{\mathbb{R}_+} \mathrm{d}(\|\mathbf{p}\|)\|\mathbf{p}\|^2 |e_\ell(\|\mathbf{p}\|)|^2$$
$$\times \int_{S^2} \mathrm{d}\widehat{\mathbf{p}}\big[p_0 + \mathbf{p}^\dagger \rho\big(k \to \Lambda_k^{-1} p\big)^\dagger \beta^*\big(-\underline{\Lambda_k^{-1} p}\big)\big]^{-1} |Y^{\mu M - j}(\hat{\mathbf{p}})|^2 \mathbb{I}_{2s+1}.$$

We have now all the elements we need to claim that

Proposition 6.4. *For $s \in \mathbb{N}/2$, $\mu \in \mathbb{N}$, $\ell \in \mathbb{Z}$, and $J \in \{|s-\mu|, \ldots, s+\mu\} \cap \{0, \ldots, s\}$, the subspaces $\mathfrak{H}_\ell^{\mu J} \subset \mathfrak{H}_w^s$ are V^J-admissible, provided the e_ℓ s are such that $r \mapsto r e_\ell(r) \in L^2(\mathbb{R}_+, r^2\,\mathrm{d}r)$.*

Proof. It remains to prove that, for all $M' = -J, -J+1, \ldots, J-1, J$,

$$0 < c_{V^J}\big(\boldsymbol{\eta}_\ell^{\mu J M'}\big) < \infty.$$

Since

$$0 < \frac{1}{2p_0} \leqslant \frac{1}{m^2}(p_0 - \|\mathbf{p}\|) \leqslant \big[p_0 + \mathbf{p}^\dagger \rho\big(k \to \Lambda_k^{-1} p\big)^\dagger \beta^*\big(-\underline{\Lambda_k^{-1} p}\big)\big]^{-1}$$
$$\leqslant \frac{1}{m^2}(p_0 + \|\mathbf{p}\|) \leqslant \frac{2}{m^2} p_0,$$

it comes from the previous lemma that, for $M' = M''$,

$$c_{V^J}\big(\boldsymbol{\eta}_\ell^{\mu J M'}\big) \leqslant \frac{(2\pi)^3}{m} \sum_{M=-J}^{J} \sum_j |C(s, j; \mu, M' - j|JM')|^2 |C(s, j; \mu, M - j|JM)|^2$$
$$\times \int_{\mathbb{R}_+} \mathrm{d}(\|\mathbf{p}\|)\|\mathbf{p}\|^2 (p_0 + \|\mathbf{p}\|)|e_\ell(\|\mathbf{p}\|)|^2$$
$$c_{V^J}\big(\boldsymbol{\eta}_\ell^{\mu J M'}\big) \geqslant \frac{(2\pi)^3}{m} \sum_{M=-J}^{J} \sum_j |C(s, j; \mu, M' - j|JM')|^2 |C(s, j; \mu, M - j|JM)|^2$$
$$\times \int_{\mathbb{R}_+} \mathrm{d}(\|\mathbf{p}\|)\|\mathbf{p}\|^2 (p_0 - \|\mathbf{p}\|)|e_\ell(\|\mathbf{p}\|)|^2 > 0.$$

If we denote by I the integral in the upper bound, and we set $r = \|\mathbf{p}\|$, we have that

$$I = \int_0^\infty \mathrm{d}r\, r^2 (\sqrt{m^2 + r^2} + r)|e_\ell(r)|^2$$
$$\leqslant 2 \int_0^\infty \mathrm{d}r\, r^2 \sqrt{m^2 + r^2} |e_\ell(r)|^2$$
$$\leqslant 2 \left[\int_0^\infty \mathrm{d}r\, r^2 |e_\ell(r)|^2\right]^{1/2} \left[\int_0^\infty \mathrm{d}r\, r^2 (m^2 + r^2)|e_\ell(r)|^2\right]^{1/2} \qquad \text{(by Schwarz)}$$
$$\leqslant 2 \left[m^2 + 2 \int_0^\infty \mathrm{d}r r^2 |r e_\ell(r)|^2\right]^{1/2} < \infty. \qquad \square$$

7. Final remarks

(i) We naturally recover the result obtained by Ali [14]. The resolution operator is not the identity (as in the case of the isochronous group of Galilei), but a non-trivial positive, self-adjoint, invertible multiplication operator. The direct consequence, as already pointed out, is that we loose the possibility of computing an orthogonality relation as one works with a representation verifying the condition (1). Nevertheless, a resolution of the identity may be obtained, for instance, with the weighted states $\widetilde{\eta}_\ell^{\mu JM}$ defined by

$$\widetilde{\eta}_\ell^{\mu JM}(k) = \mathfrak{A}_{\ell,\sigma}^{\mu J}(k)^{-1/2}\eta_\ell^{\mu JM}(k), \qquad \forall k \in \mathcal{V}_m^+. \tag{74}$$

The new fact here is that we no longer require the fiducial functions to verify the ARI. This, at least, proves that the conditions under which frames were obtained by Ali *et al* in [4, 14] are realizable in practice:

(ii) It can be easily shown that, for ℓ, μ, J fixed, the subspace $\mathfrak{H}_\ell^{\mu J}$ is proper in $\mathfrak{H}_w^s$ since its orthogonal complement contains the linear span

$$\mathrm{span}\left\{\zeta_{\ell'}^{\mu JM}, -J \leqslant M \leqslant J, \ell' \neq \ell\right\} \subset \mathfrak{H}_w^s, \tag{75}$$

where

$$\zeta_{\ell'}^{\mu JM}k = k_0^{-1}\eta_{\ell'}^{\mu JM}(k). \tag{76}$$

(iii) The assumption on the e_ℓ's is realized when they are the radial Sturmians $R_{\ell,\mu}$. Effectively, Z and a_0 denoting, respectively, the atomic number and the mass of an atom, and using the well-known results of *the virial theorem* [15], we have

$$I' = \int_0^\infty \mathrm{d}r\, r^2|re_\ell(r)|^2 = \langle r^2\rangle_{R_{\ell,\mu}} = \ell^4\left\{1 + \frac{3}{2}\left(1 - \frac{1}{\ell^2}\left[\mu(\mu+1) - \frac{1}{3}\right]\right)\right\} \tag{77}$$

in the approximation $a_0/Z \sim 1$ of the *Bohr radius*.

(iv) Let us also point out that, still in this situation, I can be related to a concrete situation in atomic physics, for example. Using the notation of Landau, since $r+\sqrt{m^2+r^2} \sim O(r)$, I can be taken as an approximation of the measure of the expectation to find a particle in a volume of radius r, and, therefore, can characterize *the size of the atom* [15].

(v) It is worth mentioning that, with the set

$$\left\{\eta_{\ell,\sigma(\mathbf{q},\mathbf{p})}^{\mu JM} : (\mathbf{q},\mathbf{p}) \in \Gamma, \ell \in \mathbb{N}^*, \mu \in \mathbb{N}, J = |s-\mu|, \ldots, s+\mu, -J \leqslant M \leqslant J\right\}, \tag{78}$$

where

$$\eta_{\ell,\sigma(\mathbf{q},\mathbf{p})}^{\mu JM}(k) = \left(\Lambda_p^{-1}k\right)_0^{1/2}\exp\{\mathrm{i}\mathbf{q}\cdot\mathbf{X}(\mathbf{k})\}\sum_j C(s,j;\mu,M-j|JM)\widehat{\mathbf{e}}^{sj}$$
$$\otimes \sum_{N=-J}^{J}\mathcal{D}^J(R(\mathbf{p}))_{NM}Y^{\mu N-j}\left(\underline{\widehat{\Lambda_p^{-1}k}}\right)e_\ell\left(\underline{\|\Lambda_p^{-1}k\|}\right), \tag{79}$$

with

$$\mathbf{X}(\mathbf{k}) = \mathbf{k} - \left(\Lambda_p^{-1}k\right)_0\vartheta(\mathbf{p}), \tag{80}$$

we get a continuous frame of rank $2J+1$, for a general space-like affine section. This frame is tight (i.e., the resolution operator is a multiple of the identity), or is not, according to the section used:

- For the Galilean section,

$$\vartheta(\mathbf{p}) = \vartheta_0(\mathbf{p}) = \mathbf{0} \quad \Leftrightarrow \quad \beta^*(\mathbf{p}) = \beta_0^*(\mathbf{p}) = \frac{\mathbf{p}}{p_0}, \tag{81}$$

  and the frame is tight only under further restrictions.

- In the case of the symmetric section,

$$\vartheta(\mathbf{p}) = \vartheta_s(\mathbf{p}) = \frac{\mathbf{p}}{m + p_0} \quad \Leftrightarrow \quad \beta^*(\mathbf{p}) = \beta_s^*(\mathbf{p}) = \frac{\mathbf{p}}{m + p_0}, \tag{82}$$

and the frame is never tight.

- The smoothest case is that of the Lorentz section, where

$$\vartheta(\mathbf{p}) = \vartheta_l(\mathbf{p}) = \frac{\mathbf{p}}{m} \quad \Leftrightarrow \quad \beta^*(\mathbf{p}) = \beta_l^*(\mathbf{p}) = \mathbf{0}, \tag{83}$$

and which leads unconditionally to a tight frame, that is,

$$\mathfrak{A}^{\mu J}_{\ell,\sigma_l} = (2\pi)^3 m(2J+1)\langle P_0^{-1}\rangle_{e_\ell}\mathbb{I}_{2s+1}. \tag{84}$$

For each ℓ, μ, J, the map $W_{\ell\mu J} : \mathfrak{H}^s_W \to \mathbb{C}^{2J+1} \otimes L^2(\Gamma, \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p})$ defined by

$$(W_{\ell\mu J}\phi)^M(\mathbf{q},\mathbf{p}) = \langle \boldsymbol{\eta}^{\mu J M}_{\ell,\sigma(\mathbf{q},\mathbf{p})} | \phi\rangle_{\mathfrak{H}^s_W}, \qquad M = -J, -J+1, \ldots, J-1, J \tag{85}$$

is an isometry. The image of $\mathfrak{H}^s_W$ in $\mathbb{C}^{2J+1} \otimes L^2(\Gamma, \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p})$ under this isometry is a reproducing kernel Hilbert space $\mathfrak{H}_{\ell\mu J}$. Its elements are vector-valued functions Φ with components

$$\Phi^M(\mathbf{q},\mathbf{p}) = \sum_j \overline{C}(s,j;\mu,M-j|JM)\sum_{N=-J}^{J}\overline{\mathcal{D}^J(R(\mathbf{p}))}_{NM}$$
$$\times \int_{\mathcal{V}_m^+} \exp\{\mathrm{i}\mathbf{q}\cdot\mathbf{X}(\mathbf{k})\}\left(\Lambda_p^{-1}k\right)_0^{1/2}\overline{Y}^{\mu N-j}\left(\widehat{\underline{\Lambda_p^{-1}k}}\right)\overline{e}_\ell\left(\left\|\underline{\Lambda_p^{-1}k}\right\|\right)\phi^j(k)\frac{\mathrm{d}\mathbf{k}}{k_0},$$
$$M = -J, -J+1, \ldots, J-1, J.$$

The matrix elements of the reproducing kernel for $\mathfrak{H}_{\ell\mu J}$ are of the form

$$\begin{aligned}&[K_{\ell\mu J}(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')]_{MM'} = \langle \boldsymbol{\eta}^{\mu J M}_{\ell,\sigma(\mathbf{q},\mathbf{p})} | \mathfrak{A}^{\mu J\,-1}_{\ell,\sigma}\boldsymbol{\eta}^{\mu J M'}_{\ell,\sigma(\mathbf{q}',\mathbf{p}')}\rangle_{\mathfrak{H}^s_W},\\ &M, M' = -J, -J+1, \ldots, J-1, J.\end{aligned} \tag{86}$$

$W_{\ell\mu J}$ can be inverted on its range to give the reconstruction formula

$$\phi = W^{-1}_{\ell\mu J}\Phi = \sum_{M=-J}^{J}\int_\Gamma \Phi^M(\mathbf{q},\mathbf{p})\mathfrak{A}^{\mu J\,-1}_{\ell,\sigma}\boldsymbol{\eta}^{\mu J M}_{\ell,\sigma(\mathbf{q},\mathbf{p})}\,\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p}. \tag{87}$$

In the case of the Galilean section σ_0, the formulae (79) and (86) particularize respectively, for $M = -J, -J+1, \ldots, J-1, J$, to

$$\boldsymbol{\eta}^{\mu J M}_{\ell,\sigma_0(\mathbf{q},\mathbf{p})}(k) = \left(\Lambda_p^{-1}k\right)_0^{1/2}\exp\{-\mathrm{i}\mathbf{q}\cdot\mathbf{k}\}$$
$$\times\sum_j C(s,j;\mu,M-j|JM)\widehat{\mathbf{e}}^{sj}\otimes Y^{\mu M-j}\left(\widehat{\underline{\Lambda_p^{-1}k}}\right)e_\ell\left(\left\|\underline{\Lambda_p^{-1}k}\right\|\right),$$

and

$$\Phi^M(\mathbf{q},\mathbf{p}) = \sum_j \overline{C}(s,j;\mu,M-j|JM)$$
$$\times\int_{\mathcal{V}_m^+}\exp\{\mathrm{i}\mathbf{q}\cdot\mathbf{k}\}\left(\Lambda_p^{-1}k\right)_0^{1/2}\overline{Y}^{\mu M-j}\left(\widehat{\underline{\Lambda_p^{-1}k}}\right)\overline{e}_\ell\left(\left\|\underline{\Lambda_p^{-1}k}\right\|\right)\phi^j(k)\frac{\mathrm{d}\mathbf{k}}{k_0}.$$

(vi) Thinking of the $\mathbf{f}^{\mu J M}_\ell$ as eigenfunctions of the Hamiltonian of some atomic system in an external potential, one could use the CS $\boldsymbol{\eta}^{\mu J M}_{\ell,\sigma(\mathbf{q},\mathbf{p})}$, $M = -J, -J+1, \ldots, J-1, J$, $(\mathbf{q},\mathbf{p}) \in \Gamma$ in analyses involving corresponding relativistic free atomic systems.

(vii) The relativistic Sturmians seem naturally adapted for the analysis of $SU(2)$-symmetry-invariant systems like hydrogenic atoms.

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Appendix A. Proof of lemma 6.1

Proof. For $R \in K$, we have that

$$
\begin{aligned}
\left[U_w^s(0,R)\eta_\ell^{\mu JM}\right](k) &= \mathcal{D}^s(w(k,R))\eta_\ell^{\mu JM}(L(R)^{-1}k) \\
&= \mathcal{D}^s(R)(L(R)^{-1}k)_0^{1/2}\mathbf{f}_\ell^{\mu JM}(\underline{L(R)^{-1}k}) = k_0^{1/2}\mathcal{D}^s(R)\mathbf{f}_\ell^{\mu JM}(\rho^{-1}\mathbf{k}) \\
&\quad \left(\text{because } L(R) = \begin{pmatrix} 1 & \mathbf{0}^\dagger \\ \mathbf{0} & \rho \end{pmatrix},\ \text{with } \rho \in SO(3),\ \text{and } w(k,R) = R\right) \\
&= k_0^{1/2}\mathcal{D}^s(R)\sum_{\nu+\tau=M} C(s,\nu;\mu,\tau|JM)\hat{\mathbf{e}}^{s\nu} \otimes Y^{\mu\tau}(\rho^{-1}\hat{\mathbf{k}})e_\ell(\|\rho^{-1}\mathbf{k}\|) \\
&= k_0^{1/2}\sum_{\nu+\tau=M} C(s,\nu;\mu,\tau|JM)\left[\sum_{\epsilon=-s}^{s}\mathcal{D}^s(R)_{\epsilon\nu}\hat{\mathbf{e}}^{s\epsilon}\right] \\
&\quad \otimes \left[\sum_{\lambda=-\mu}^{\mu}\mathcal{D}^\mu(R)_{\lambda\tau}Y^{\mu\lambda}(\hat{\mathbf{k}})\right]e_\ell(\|\mathbf{k}\|) \\
&= \quad k_0^{1/2}\sum_{\nu+\tau=M} C(s,\nu;\mu,\tau|JM)\sum_{\epsilon=-s}^{s}\hat{\mathbf{e}}^{s\epsilon} \\
&\quad \otimes \sum_{\lambda=-\mu}^{\mu}\mathcal{D}^s(R)_{\epsilon\nu}\mathcal{D}^\mu(R)_{\lambda\tau}Y^{\mu\lambda}(\hat{\mathbf{k}})e_\ell(\|\mathbf{k}\|) \\
&= k_0^{1/2}\sum_{L=|s-\mu|}^{s+\mu}\sum_{N,Q=-L}^{L}\mathcal{D}^L(R)_{NQ}\sum_{\epsilon+\lambda=N}C(s,\epsilon;\mu,\lambda|LN) \\
&\quad \times \sum_{\nu+\tau=M}C(s,\nu;\mu,\tau|JM)C(s,\nu;\mu,\tau|LQ)\hat{\mathbf{e}}^{s\epsilon} \otimes Y^{\mu\lambda}(\hat{\mathbf{k}})e_\ell(\|\mathbf{k}\|) \\
&\quad (\text{because } \mathcal{D}^s(R)_{\epsilon\nu}\mathcal{D}^\mu(R)_{\lambda\tau} \\
&= \sum_{L=|s-\mu|}^{s+\mu}\sum_{N,Q=-L}^{L}C(s,\epsilon;\mu,\lambda|LN)\mathcal{D}^L(R)_{NQ}C(s,\nu;\mu,\tau|LQ)) \\
&= k_0^{1/2}\sum_{L=|s-\mu|}^{s+\mu}\delta_{JL}\sum_{N,Q=-L}^{L}\mathcal{D}^L(R)_{NQ}\delta_{MQ} \\
&\quad \times \sum_{\epsilon+\lambda=N}C(s,\epsilon;\mu,\lambda|LN)\hat{\mathbf{e}}^{s\epsilon} \otimes Y^{\mu\lambda}(\hat{\mathbf{k}})e_\ell(\|\mathbf{k}\|) \\
&\quad \left(\text{since } \sum_{\nu+\tau=M}C(s,\nu;\mu,\tau|JM)C(s,\nu;\mu,\tau|LQ) = \delta_{JL}\delta_{MQ}\right) \\
&= k_0^{1/2}\sum_{L=|s-\mu|}^{s+\mu}\delta_{JL}\sum_{N,Q=-L}^{L}\mathcal{D}^L(R)_{NQ}\delta_{MQ}\mathbf{f}_\ell^{\mu LN}(\mathbf{k}) \\
&= k_0^{1/2}\sum_{N,Q=-J}^{J}\mathcal{D}^J(R)_{NQ}\delta_{MQ}\mathbf{f}_\ell^{\mu JN}(\mathbf{k})
\end{aligned}
$$

$$
\begin{aligned}
&= k_0^{1/2} \sum_{N=-J}^{J} \mathcal{D}^J(R)_{NM} \mathbf{f}_\ell^{\mu J N}(\mathbf{k}) \\
&= k_0^{1/2} \big[\mathcal{D}^J(R) \mathbf{f}_\ell^{\mu J M}\big](\mathbf{k}) \\
&= \big[\mathcal{F}^{-1} \mathcal{D}^J(R) \mathcal{F} \boldsymbol{\eta}_\ell^{\mu J M}\big](k). \qquad \square
\end{aligned}
$$

Appendix B. Proof of lemma 6.2

Proof. We have that

$$
\begin{aligned}
\big[U_w^s(0,B)\boldsymbol{\eta}_\ell^{\mu J M}\big](k) &= \mathcal{D}^s(w(k,B))\boldsymbol{\eta}_\ell^{\mu J M}(L(B)^{-1}k) \\
&= (L(B)^{-1}k)_0^{1/2} \mathcal{D}^s(w(k,B)) \mathbf{f}_\ell^{\mu J M}(\underline{L(B)^{-1}k}) \\
&= (L(B)^{-1}k)_0^{1/2} \sum_{\nu+\tau=M} C(s,\nu;\mu,\tau|JM) \left[\sum_{\epsilon=-s}^{s} \mathcal{D}^s(w(k,B))_{\epsilon\nu} \hat{\mathbf{e}}^{s\epsilon}\right] \\
&\quad \otimes \left[\sum_{\lambda=-\mu}^{\mu} \mathcal{D}^\mu(w(k,B))_{\lambda\tau} Y^{\mu\lambda}(\varpi' \underline{L(B)^{-1}k})\right] e_\ell(\varpi' \underline{L(B)^{-1}k}) \\
&\quad (\varpi' \text{ is the image in } SO(3) \text{ of } w(k,B) \in SU(2)) \\
&= (L(B)^{-1}k)_0^{1/2} \sum_{N=-J}^{J} \mathcal{D}^J(w(k,B))_{NM} \mathbf{f}_\ell^{\mu J N}(\varpi' \underline{L(B)^{-1}k}).
\end{aligned}
$$

Using the well-known action of rotations, we can write that

$$
\big[U_w^s(0,B)\boldsymbol{\eta}_\ell^{\mu J M}\big](k) = (L(B)^{-1}k)_0^{1/2} \mathbf{f}_\ell^{\mu J N}(\underline{L(B)^{-1}k}) = \boldsymbol{\eta}_\ell^{\mu J M}(L(B)^{-1}k).
$$

On the other hand, we have that

$$
\begin{aligned}
\big[U_w^s(0,B)\boldsymbol{\eta}_\ell^{\mu J M}\big](k) &= (L(B)^{-1}k)_0^{1/2} \big[\mathcal{D}^J(w(k,B)) \mathbf{f}_\ell^{\mu J M}\big](\varpi' \underline{L(B)^{-1}k}) \\
&= \big[\mathcal{F}^{-1} \mathcal{D}^J(w(k,B)) \mathcal{F} \boldsymbol{\eta}_\ell^{\mu J M}\big](L(w(k,B))L(B)^{-1}k). \qquad \square
\end{aligned}
$$

Appendix C. Proof of lemma 6.3

Proof. Let us remark that

$$
\sigma(\mathbf{q},\mathbf{p}) = (\widehat{q}, S(p)R(\mathbf{p})) = (\widehat{q}, S(p))(0, R(\mathbf{p})),
$$

$$
\begin{aligned}
\big\langle \boldsymbol{\eta}_\ell^{\mu J M'} \big| \boldsymbol{\eta}_{\ell,\sigma(\mathbf{q},\mathbf{p})}^{\mu J M} \big\rangle_{\mathfrak{H}_W^s} &= \big\langle U_w^s((\widehat{q}, S(p)))^\dagger \boldsymbol{\eta}_\ell^{\mu J M'} \big| U_w^s((0, R(\mathbf{p})))\boldsymbol{\eta}_\ell^{\mu J M} \big\rangle_{\mathfrak{H}_W^s} \\
&= \big\langle U_w^s((\hat{q}, S(p)))^\dagger \boldsymbol{\eta}_\ell^{\mu J M'} \big| \mathcal{F}^{-1}\mathcal{D}^J(R(\mathbf{p}))\mathcal{F}\boldsymbol{\eta}_\ell^{\mu J M} \big\rangle_{\mathfrak{H}_W^s} \\
&= \big\langle \mathcal{F} U_w^s((\widehat{q}, S(p)))^\dagger \boldsymbol{\eta}_\ell^{\mu J M'} \big| \mathcal{D}^J(R(\mathbf{p}))\mathbf{f}_\ell^{\mu J M} \big\rangle_{\mathcal{H}^s},
\end{aligned}
$$

and

$$
\begin{aligned}
\big\langle \mathbf{f}_\ell^{\mu J M'}(\mathbf{k}) \big| \mathbf{f}_\ell^{\mu J M}(\mathbf{p}) \big\rangle_{2J+1} &= e_\ell(\|\mathbf{k}\|) e_\ell(\|\mathbf{p}\|) \sum_\nu \overline{C}(s,\nu;\mu,M'-\nu|JM') \\
&\quad \times C(s,\nu;\mu,M-\nu|JM) \overline{Y}^{\mu M'-\nu}(\widehat{\mathbf{k}}) Y^{\mu M-\nu}(\widehat{\mathbf{p}}).
\end{aligned}
$$

Then,

$$
\begin{aligned}
c_{V^J}\left(\boldsymbol{\eta}_\ell^{\mu JM'},\boldsymbol{\eta}_\ell^{\mu JM''}\right) &= \sum_{M=-J}^{J}\int_\Gamma \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p}\langle\boldsymbol{\eta}_\ell^{\mu JM'}\big|\boldsymbol{\eta}_{\ell,\sigma(\mathbf{q},\mathbf{p})}^{\mu JM}\big\rangle_{\mathfrak{H}_W^s}\big\langle\boldsymbol{\eta}_{\ell,\sigma(\mathbf{q},\mathbf{p})}^{\mu JM}\big|\boldsymbol{\eta}_\ell^{\mu JM''}\big\rangle_{\mathfrak{H}_W^s} \\
&= \sum_{M=-J}^{J}\int_\Gamma \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p}\big\langle\mathcal{F}U_w^s((\widehat{q},S(p)))^\dagger\boldsymbol{\eta}_\ell^{\mu JM'}\big|\mathcal{D}^J(R(\mathbf{p}))\mathbf{f}_\ell^{\mu JM}\big\rangle_{\mathcal{H}^s} \\
&\quad\times\big\langle\mathcal{D}^J(R(\mathbf{p}))\mathbf{f}_\ell^{\mu JM}\big|\mathcal{F}U_w^s((\widehat{q},S(p)))^\dagger\boldsymbol{\eta}_\ell^{\mu JM''}\big\rangle_{\mathcal{H}^s} \\
&= \sum_{M=-J}^{J}\int_\Gamma \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p}\big\langle\mathcal{F}U_w^s((\widehat{q},S(p)))^\dagger\boldsymbol{\eta}_\ell^{\mu JM'}\big|\mathbf{f}_\ell^{\mu JM}\big\rangle_{\mathcal{H}^s} \\
&\quad\times\big\langle\mathbf{f}_\ell^{\mu JM}\big|\mathcal{F}U_w^s((\widehat{q},S(p)))^\dagger\boldsymbol{\eta}_\ell^{\mu JM''}\big\rangle_{\mathcal{H}^s} \\
&= \sum_{M=-J}^{J}\int_\Gamma \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p}\big\langle\boldsymbol{\eta}_\ell^{\mu JM'}\big|U_w^s((\widehat{q},S(p)))\boldsymbol{\eta}_\ell^{\mu JM}\big\rangle_{\mathfrak{H}_W^s} \\
&\quad\times\big\langle U_w^s((\widehat{q},S(p)))\boldsymbol{\eta}_\ell^{\mu JM}\big|\boldsymbol{\eta}_\ell^{\mu JM''}\big\rangle_{\mathfrak{H}_W^s} \\
&= \sum_{M=-J}^{J}\int_{\Gamma\times\mathcal{V}_m^+\times\mathcal{V}_m^+}\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p}\frac{\mathrm{d}\mathbf{k}}{k_0}\frac{\mathrm{d}\mathbf{k}'}{k_0'}\exp\{-\mathrm{i}[\mathbf{X}(\mathbf{k})-\mathbf{X}(\mathbf{k}')]\cdot\mathbf{q}\} \\
&\quad\times\big\langle\boldsymbol{\eta}_\ell^{\mu JM'}(k)\big|\big[U_w^s((0,S(p)))\boldsymbol{\eta}_\ell^{\mu JM}\big](k)\big\rangle_{2s+1} \\
&\quad\times\big\langle\big[U_w^s((0,S(p)))\boldsymbol{\eta}_\ell^{\mu JM}\big](k')\big|\boldsymbol{\eta}_\ell^{\mu JM''}(k')\big\rangle_{2s+1},
\end{aligned}
$$

with $\mathbf{X}(\mathbf{k})=\mathbf{k}-\left(\Lambda_p^{-1}k\right)_0\boldsymbol{\vartheta}(\mathbf{p})$. Doing then the change $\mathbf{k}\mapsto\mathbf{X}(\mathbf{k})$ with the assumption (26), we perform the integration over $\mathbf{q}$ and, when we come back to the variable k and apply the result of lemma 6.2, we get

$$
\begin{aligned}
c_{V^J}\left(\boldsymbol{\eta}_\ell^{\mu JM'},\boldsymbol{\eta}_\ell^{\mu JM''}\right) &= (2\pi)^3 m\sum_{M=-J}^{J}\int_{\mathcal{V}_m^+\times\mathcal{V}_m^+}\frac{\mathrm{d}\mathbf{k}}{k_0}\frac{\mathrm{d}\mathbf{p}}{p_0}\left[\left(\Lambda_p^{-1}k\right)_0+\left(\underline{\Lambda_p^{-1}k}\right)^\dagger\boldsymbol{\beta}^*(\mathbf{p})\right]^{-1} \\
&\quad\times\big\langle\boldsymbol{\eta}_\ell^{\mu JM'}(k)\big|\boldsymbol{\eta}_\ell^{\mu JM}\left(\Lambda_p^{-1}k\right)\big\rangle_{2s+1}\big\langle\boldsymbol{\eta}_\ell^{\mu JM}\left(\Lambda_p^{-1}k\right)\big|\boldsymbol{\eta}_\ell^{\mu JM''}(k)\big\rangle_{2s+1} \\
&= (2\pi)^3 m\sum_{M=-J}^{J}\int_{\mathcal{V}_m^+\times\mathcal{V}_m^+}\frac{\mathrm{d}\mathbf{k}}{k_0}\frac{\mathrm{d}\mathbf{p}}{p_0}[(\Lambda_k\overline{p})_0+(\underline{\Lambda_k\overline{p}})^\dagger\rho(k\to\overline{p})^\dagger\boldsymbol{\beta}^*(\mathbf{p})]^{-1} \\
&\quad\times\big\langle\boldsymbol{\eta}_\ell^{\mu JM'}(k)\big|\boldsymbol{\eta}_\ell^{\mu JM}(\rho(k\to\overline{p})\Lambda_k\overline{p})\big\rangle_{2s+1} \\
&\quad\times\big\langle\boldsymbol{\eta}_\ell^{\mu JM}(\rho(k\to\overline{p})\Lambda_k\overline{p})\big|\boldsymbol{\eta}_\ell^{\mu JM''}(k)\big\rangle_{2s+1} \\
&\qquad(\text{because } \Lambda_p^{-1}k=\rho(k\to\overline{p})\Lambda_k\overline{p}) \\
&= (2\pi)^3 m\sum_{M=-J}^{J}\int_{\mathcal{V}_m^+\times\mathcal{V}_m^+}\frac{\mathrm{d}\mathbf{k}}{k_0}\frac{\mathrm{d}\mathbf{p}}{p_0}\left[p_0+\mathbf{p}^\dagger\rho\left(k\to\Lambda_k^{-1}p\right)^\dagger\boldsymbol{\beta}^*\left(-\underline{\Lambda_k^{-1}p}\right)\right]^{-1} \\
&\quad\times\big\langle\boldsymbol{\eta}_\ell^{\mu JM'}(k)\big|\boldsymbol{\eta}_\ell^{\mu JM}\left(\rho\left(k\to\Lambda_k^{-1}p\right)p\right)\big\rangle_{2s+1} \\
&\quad\times\big\langle\boldsymbol{\eta}_\ell^{\mu JM}\left(\rho\left(k\to\Lambda_k^{-1}p\right)p\right)\big|\boldsymbol{\eta}_\ell^{\mu JM''}(k)\big\rangle_{2s+1} \\
&\quad(\text{by the change } \overline{p}\mapsto\Lambda_k^{-1}p,\ \text{and measure invariance}) \\
&= (2\pi)^3 m\sum_{M=-J}^{J}\int_{\mathbb{R}^3\times\mathbb{R}^3}\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{p}\left[p_0+\mathbf{p}^\dagger\rho\left(k\to\Lambda_k^{-1}p\right)^\dagger\boldsymbol{\beta}^*\left(-\underline{\Lambda_k^{-1}p}\right)\right]^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\times \langle \mathbf{f}_\ell^{\mu J M'}(\mathbf{k}) | \mathbf{f}_\ell^{\mu J M}\big(R\big(k \to \Lambda_k^{-1} p\big)\mathbf{p}\big)\rangle_{2s+1} \\
&\times \langle \mathbf{f}_\ell^{\mu J M}\big(R\big(k \to \Lambda_k^{-1} p\big)\mathbf{p}\big) | \mathbf{f}_\ell^{\mu J M''}(\mathbf{k})\rangle_{2s+1} \\
&= (2\pi)^3 m \sum_{M=-J}^{J} \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{p}\big[p_0 + \mathbf{p}^\dagger \rho\big(k \to \Lambda_k^{-1} p\big)^\dagger \beta^*\big(-\underline{\Lambda_k^{-1} p}\big)\big]^{-1} \\
&\times \sum_{N=-\mu}^{\mu} \sum_{Q=-\mu}^{\mu} \mathcal{D}^\mu\big(R\big(k \to \Lambda_k^{-1} p\big)^{-1}\big)_{NM} \overline{\mathcal{D}^\mu\big(R\big(k \to \Lambda_k^{-1} p\big)^{-1}\big)_{QM}} \\
&\times \langle \mathbf{f}_\ell^{\mu J M'}(\mathbf{k}) | \mathbf{f}_\ell^{\mu J N}(\mathbf{p})\rangle_{2s+1} \langle \mathbf{f}_\ell^{\mu J Q}(\mathbf{p}) | \mathbf{f}_\ell^{\mu J M''}(\mathbf{k})\rangle_{2s+1} \\
&= (2\pi)^3 m \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{p}\big[p_0 + \mathbf{p}^\dagger \rho\big(k \to \Lambda_k^{-1} p\big)^\dagger \beta^*\big(-\underline{\Lambda_k^{-1} p}\big)\big]^{-1} \\
&\times \sum_{N=-\mu}^{\mu} \sum_{Q=-\mu}^{\mu} \delta_{NQ} \langle \mathbf{f}_\ell^{\mu J M'}(\mathbf{k}) | \mathbf{f}_\ell^{\mu J N}(\mathbf{p})\rangle_{2s+1} \langle \mathbf{f}_\ell^{\mu J Q}(\mathbf{p}) | \mathbf{f}_\ell^{\mu J M''}(\mathbf{k})\rangle_{2s+1} \\
&= (2\pi)^3 m \sum_{N=-\mu}^{\mu} \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{p}\big[p_0 + \mathbf{p}^\dagger \rho\big(k \to \Lambda_k^{-1} p\big)^\dagger \beta^*\big(-\underline{\Lambda_k^{-1} p}\big)\big]^{-1} \\
&\times \langle \mathbf{f}_\ell^{\mu J M'}(\mathbf{k}) | \mathbf{f}_\ell^{\mu J N}(\mathbf{p})\rangle_{2s+1} \langle \mathbf{f}_\ell^{\mu J N}(\mathbf{p}) | \mathbf{f}_\ell^{\mu J M''}(\mathbf{k})\rangle_{2s+1} \\
&= (2\pi)^3 m \sum_{M=-J}^{J} \int_{\mathcal{V}_m^+\times\mathcal{V}_m^+} \frac{\mathrm{d}\mathbf{k}}{k_0}\frac{\mathrm{d}\mathbf{p}}{p_0}\big[p_0 + \mathbf{p}^\dagger \rho\big(k \to \Lambda_k^{-1} p\big)^\dagger \beta^*\big(-\underline{\Lambda_k^{-1} p}\big)\big]^{-1} \\
&\times \langle \boldsymbol{\eta}_\ell^{\mu J M'}(k) | \boldsymbol{\eta}_\ell^{\mu J M}(p)\rangle_{2s+1} \langle \boldsymbol{\eta}_\ell^{\mu J M}(p) | \boldsymbol{\eta}_\ell^{\mu J M''}(k)\rangle_{2s+1} \\
&= \int_{\mathcal{V}_m^+} \frac{\mathrm{d}\mathbf{k}}{k_0} \langle \boldsymbol{\eta}_\ell^{\mu J M'}(k) | \mathfrak{A}_{\ell,\sigma}^{\mu J}(k) \boldsymbol{\eta}_\ell^{\mu J M''}(k)\rangle_{2s+1} = \langle \boldsymbol{\eta}_\ell^{\mu J M'} | \mathfrak{A}_{\ell,\sigma}^{\mu J} \boldsymbol{\eta}_\ell^{\mu J M''}\rangle_{\mathfrak{H}_W^s},
\end{aligned}
$$

with

$$
\begin{aligned}
\mathfrak{A}_{\ell,\sigma}^{\mu J}(k) =& (2\pi)^3 m \int_{\mathcal{V}_m^+} \frac{\mathrm{d}\mathbf{p}}{p_0}\big[p_0 + \mathbf{p}^\dagger \rho\big(k \to \Lambda_k^{-1} p\big)^\dagger \beta^*\big(-\underline{\Lambda_k^{-1} p}\big)\big]^{-1} \\
&\times \sum_{M=-J}^{J} \big|\boldsymbol{\eta}_\ell^{\mu J M}(p)\big\rangle\big\langle \boldsymbol{\eta}_\ell^{\mu J M}(p)\big| \\
=& (2\pi)^3 m \sum_{M=-J}^{J} \sum_j |C(s, j; \mu, M-j|JM)|^2 \int_{\mathbb{R}_+} \mathrm{d}(\|\mathbf{p}\|)\|\mathbf{p}\|^2 |e_\ell(\|\mathbf{p}\|)|^2 \\
&\times \int_{S^2} \mathrm{d}\widehat{\mathbf{p}}\big[p_0 + \mathbf{p}^\dagger \rho\big(k \to \Lambda_k^{-1} p\big)^\dagger \beta^*\big(-\underline{\Lambda_k^{-1} p}\big)\big]^{-1} |Y^{\mu M-j}(\widehat{\mathbf{p}})|^2 \mathbb{I}_{2s+1}. \qquad \square
\end{aligned}
$$

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